

# Note on the location of zeros of polynomials

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## Abstract

In this note, we provide a wide range of upper bounds for the moduli of the zeros of a complex polynomial. The obtained bounds complete a series of previous papers on the location of zeros of polynomials.

*Keywords:* Complex polynomials; Location of zeros of polynomials; Cauchy’s bound

*2000MSC:* 26C10, 30C15, 65H05

## 1 Introduction

The theory of the location of zeros of polynomials has applications in several areas of contemporary applied mathematics, including linear control systems, electrical networks, root approximation, signal processing and coding theory. Because of its applications, there is a need for obtaining better and better results in this subject. A review on the location of zeros of polynomials can be found in [1, 2, 3].

In what follows,  $P(z)$  is the complex polynomial

$$P(z) = z^n + a_1 z^{n-1} + \cdots + a_{n-1} z + a_n. \quad (1)$$

Without loss of generality we will assume that  $a_j \neq 0$  for at least one  $j$ , and set  $a_j = 0$  for  $j > n$ . According to a result of Cauchy [4], all the zeros of the polynomial  $P(z)$  are in the circle  $|z| \leq \rho$ , where  $\rho$  is the unique positive zero of the real polynomial

$$Q(x) = x^n - |a_1| x^{n-1} - \cdots - |a_{n-1}| x - |a_n|. \quad (2)$$

The upper bound  $\rho$  is the best possible one which is expressible in terms of the moduli of the coefficients. A classical result due to Cauchy [4] states that all the zeros of the polynomial  $P(z)$  are contained in the disk

$$|z| \leq \rho < 1 + A \quad (3)$$

with  $A = \max_{1 \leq j \leq n} |a_j|$ . As an improvement, Joyal, Labelle and Rahman [5] proved the following theorem.

**Theorem 1** *All the zeros of  $P(z)$  are contained in the disk*

$$|z| \leq \rho \leq \frac{1}{2}(|a_1| + 1 + \sqrt{(|a_1| - 1)^2 + 4A_2}) \quad (4)$$

with  $A_2 = \max_{2 \leq j \leq n} |a_j|$ .

For each positive integer number  $\ell \geq 1$ , let  $Q_\ell(x)$  be the polynomial

$$Q_\ell(x) = x^\ell + \sum_{v=2}^{\ell} \left[ C_{\ell-v}^{\ell-1} - \sum_{j=1}^{v-1} C_{\ell-v}^{\ell-j-1} |a_j| \right] x^{\ell+1-v}, \quad (5)$$

where  $C_s^m$  ( $0 \leq s \leq m$ ) are the binomial coefficients defined by  $C_s^m = \frac{m!}{s!(m-s)!}$ . More recently, Affane–Aji et al. [6] have obtained the following result.

**Theorem 2** *All the zeros of  $P(z)$  satisfy  $|z| < 1 + \delta_\ell$ , for  $\ell = 1, 2, \dots$ , where  $\delta_\ell$  (for  $\ell$  a positive integer) is the unique positive root of the  $\ell$ th degree equation*

$$Q_\ell(x) = A. \quad (6)$$

Moreover,  $1 + \delta_1 \geq 1 + \delta_2 \geq \dots \geq 1 + \delta_\ell > \max(1, \rho)$ , for all  $\ell \geq 1$ .

Theorem 2 provides a tool for obtaining sharper bounds for the location of the zeros of a polynomial. When  $\ell = 1$  it reduces to (3), and for  $\ell = 2$  it yields

$$|z| < 1 + \delta_2 = \frac{1}{2}(|a_1| + 1 + \sqrt{(|a_1| - 1)^2 + 4A}), \quad (7)$$

which looks like (4) but never sharpens it. The cases  $\ell = 3$  and  $\ell = 4$  give rise to cubic and quartic equations which can be explicitly solved, and they are due to Sun and Hsieh [7] and Jain [8] respectively.

Observe that there is no way to establish a link between Theorem 1 and Theorem 2, except for the case  $\ell = 2$ , in which Theorem 1 provides a better bound. For example, if  $P(z) = z^n + az^{n-1}$  with  $|a| > 0$ , then Theorem 1 yields  $|z| \leq \max(1, |a|)$ . But  $\rho = |a|$ , so  $1 + \delta_\ell > \max(1, |a|)$  for any  $\ell$ . Hence, in this case, the bound obtained from Theorem 1 is better than any bound obtained from Theorem 2, although the last ones may require a high computational cost in order to be obtained.

The aim of this note is to fill this gap by showing that Theorem 2 can be sharpened in a natural way, so that it contains Theorem 1 as a particular case (see Theorem 3). Numerical examples will show that our improvement provides bounds which may be considerably better than the preceding ones.

The main result, Theorem 3, completes the series of papers [6, 7, 8] on the location of zeros of polynomials, and fills the gaps between them and Theorem 1, due to Joyal et al. [5].

## 2 The main result

In what follows, we denote by  $\varepsilon_\ell$  the largest real root of the  $\ell$ th degree equation

$$Q_\ell(x) = A_\ell \quad (8)$$

with  $A_\ell = \max_{j \geq \ell} |a_j|$ ,  $\ell \geq 1$ , ( $A_\ell = 0$  for  $\ell > n$ ). Additionally,  $q \in \{1, \dots, n\}$  is defined by the conditions

$$a_q \neq 0, \quad \text{and } a_j = 0 \quad \text{for } j > q. \quad (9)$$

**Theorem 3** *We have*

$$1 + \varepsilon_1 \geq 1 + \varepsilon_2 \geq \dots \geq 1 + \varepsilon_q > \max(1, \rho) = 1 + \varepsilon_{q+1} = 1 + \varepsilon_{q+2} = \dots$$

*In particular, all the zeros of the polynomial  $P(z)$  satisfy  $|z| < 1 + \varepsilon_\ell$ , for  $\ell = 1, \dots, q$ , and  $|z| \leq 1 + \varepsilon_\ell$  for  $\ell > q$ . Furthermore,  $\varepsilon_\ell$  (for  $1 \leq \ell \leq q$ ) is the unique positive root of the equation  $Q_\ell(x) = A_\ell$ .*

Observe that when  $\ell = 1$ , the bound obtained from this result reduces to (3), as Theorem 2 did. However, for  $\ell = 2$  now we obtain the bound (4), thus meeting Theorem 1 as desired. Finally, if  $\ell > q$ , we have  $|z| \leq 1 + \varepsilon_\ell = \max(1, \rho)$ , which reduces to  $|z| \leq \rho$  when  $\rho \geq 1$ .

Next, we prove that Theorem 3 sharpens Theorem 2. To this end, we shall show that  $\varepsilon_\ell \leq \delta_\ell$  for any  $\ell$ . Indeed, if  $\ell > q$  then  $1 + \varepsilon_\ell = \max(1, \rho) < 1 + \delta_\ell$ , hence  $\varepsilon_\ell < \delta_\ell$ . For  $1 \leq \ell \leq q$ , we consider the polynomial  $R(x) = Q_\ell(x) - A_\ell$ . Since  $R(0) = -A_\ell < 0$  then  $R(x) < 0$  for  $0 < x < \varepsilon_\ell$ , and taking into account that  $R(\delta_\ell) = A - A_\ell \geq 0$ , we obtain  $\delta_\ell \geq \varepsilon_\ell$ , and we are done. Observe that  $\varepsilon_\ell < \delta_\ell$  when  $A_\ell < A$ .

**Remark 4** *By looking at equations (6) and (8), the algorithm using MATLAB that has been successfully developed in [6], and which has as its output the upper bound  $1 + \delta_\ell$ , may be used to obtain an algorithm having as its output the upper bound  $1 + \varepsilon_\ell$ , by simply replacing number  $A$  in that algorithm by number  $A_\ell$ . This simple modification should be taken under consideration, since the bounds obtained from Theorem 3 may be considerably better than the ones obtained from Theorem 2.*

**Example 5** *For the polynomial  $P(z) = z^5 + 3z^4 + 2z^2 + 2$ , we have  $\rho = 3.21256$  and it coincides*

with the largest modulus of the zeros. On the other hand, we have

$\ell$	$1 + \varepsilon_\ell$	$1 + \delta_\ell$
1	4.00000	4.00000
2	3.73205	4.00000
3	3.26953	3.37442
4	3.26953	3.30278
5	3.21989	3.23138
6	3.21256( $= \rho$ )	3.22350

**Example 6** Let  $P(z) = z^{10} + 2z^9 - 3z^8 + 2z^5 - z^4 + z + 2$ . Then  $\rho = 3.02120$  and the largest modulus of the zeros is 3.02106. We have

$\ell$	$1 + \varepsilon_\ell$	$1 + \delta_\ell$
1	4.00000	4.00000
2	3.30278	3.30278
3	3.21432	3.30278
4	3.07678	3.11111
5	3.02675	3.03942
10	3.02124	3.02129
11	3.02120	3.02125

**Example 7** For the polynomial  $P(z) = z^{20} - 0.6z^{19} - 0.3z^{15} - 0.2z^8 - 0.1z - 0.2$ , we have  $\rho = 1.05673$ , which coincides with the largest modulus of the zeros. The bounds are

$\ell$	$1 + \varepsilon_\ell$	$1 + \delta_\ell$
1	1.60000	1.60000
2	1.38310	1.60000
3	1.31742	1.46954
4	1.27413	1.39150
5	1.24297	1.33864
6	1.20500	1.31930
10	1.15805	1.22986
21	1.05673	1.14649

### 3 Proof of the main result

In order to prove Theorem 3, first we will prove a result that will be shown to be equivalent to the main result, and which is interesting in itself because it simplifies considerably the expression of the equation to be solved.

For  $\ell \geq 2$  an integer number, let  $r_\ell$  be the largest real zero of the  $\ell$ th degree polynomial

$$P_\ell(x) = x^\ell - (|a_1| + 1)x^{\ell-1} - \sum_{j=2}^{\ell-1} (|a_j| - |a_{j-1}|)x^{\ell-j} - (A_\ell - |a_{\ell-1}|), \quad (10)$$

where  $a_j = 0$  for  $j > n$ . When  $\ell = 1$  we define  $r_1 = 1 + A$ , the unique zero of the polynomial  $P_1(x) = x - (1 + A)$ . In what follows,  $q$  is defined by the conditions (9). Then,

**Theorem 8** *We have*

$$r_1 \geq r_2 \geq \cdots \geq r_q > \max(1, \rho) = r_{q+1} = r_{q+2} = \cdots \quad (11)$$

*In particular, all the zeros of the polynomial  $P(z)$  satisfy  $|z| < r_\ell$ , for  $\ell = 1, \dots, q$ , and  $|z| \leq r_\ell$  for  $\ell > q$ . Furthermore,  $r_\ell$  (for  $1 \leq \ell \leq q$ ) is the unique zero of the polynomial  $P_\ell(x)$  in the interval  $[1, +\infty)$ .*

*Proof.* By dividing the polynomial  $P_\ell(x)$  by  $x - 1$ , we have

$$P_\ell(x) = (x - 1)F_\ell(x) - A_\ell, \quad (12)$$

where

$$F_\ell(x) = x^{\ell-1} - |a_1|x^{\ell-2} - \cdots - |a_{\ell-2}|x - |a_{\ell-1}|, \quad \text{for } \ell \geq 2, \quad (13)$$

and  $F_1(x) = 1$ .

We need a lemma which is part of the statement of Theorem 8.

**Lemma 9** *If  $1 \leq \ell \leq q$ , then  $P_\ell(x)$  has a unique zero in  $[1, +\infty)$ .*

*Proof.* Let  $\ell \in \{1, 2, \dots, q\}$  be given. Since  $P_\ell(1) = -A_\ell < 0$  and  $P_\ell(x)$  tends to  $+\infty$  when  $x$  tends to  $+\infty$ , there exists at least one zero of  $P_\ell(x)$  in  $[1, +\infty)$ . Let  $\alpha$  be the largest real zero of  $F_\ell(x)$  (for  $\ell = 1$  we set  $\alpha = 0$ ). By Descartes's rule of signs, if some coefficient  $a_j$  for  $1 \leq j \leq \ell - 1$  is nonzero, then  $\alpha$  is the unique strictly positive zero of  $F_\ell(x)$ ; otherwise  $\alpha = 0$ .

Set  $\mu = \max(1, \alpha)$ , and we claim that  $P_\ell(x)$  is strictly increasing for  $x \geq \mu$ . In fact, if  $\alpha = 0$  (i.e.  $F_\ell(x)$  has the form  $F_\ell(x) = x^{\ell-1}$ ), then  $P_\ell(x) = (x - 1)x^{\ell-1} - A_\ell$ , an increasing function for  $x \geq 1 = \mu$ . Assume that  $\alpha > 0$ , and let

$$F_\ell(x) \equiv (x - \alpha)S(x)$$

with  $S(x)$  a real monic polynomial. From  $\alpha > 0$  and the fact that the sequence of nonzero coefficients of  $F_\ell(x)$  has only one change of sign, it follows that all the coefficients of  $S(x)$  must be nonnegative; in particular,  $S(x)$  is monotonically increasing and positive for  $x \geq 0$ . This implies that  $P_\ell(x)$  is strictly increasing for  $x \geq \mu$ , as

$$P_\ell(x) \equiv (x - 1)(x - \alpha)S(x) - A_\ell.$$

The claim is proved.

Since  $P_\ell(\mu) = -A_\ell < 0$  and  $P_\ell(x)$  is strictly increasing for  $x \geq \mu$ ,  $P_\ell(x)$  has a unique zero in the interval  $[\mu, +\infty)$ . Therefore, when  $\mu = 1$  it is the unique zero in  $[1, +\infty)$ , and we are done. Finally, assume that  $\mu = \alpha > 1$ , and we shall see that  $P_\ell$  is zero-free in the interval  $[1, \mu)$ . Indeed, if  $1 \leq y < \mu = \alpha$  then  $F_\ell(y) < 0$  and  $P_\ell(y) = (y-1)F_\ell(y) - A_\ell \leq -A_\ell < 0$ , hence  $P_\ell(y) \neq 0$  for  $y \in [1, \mu)$ , as desired. This completes the proof of the lemma.  $\square$

Taking into account the preceding lemma and that  $\rho$  is an upper bound for the moduli of the zeros of  $P(z)$ , in order to prove Theorem 8 it suffices to prove the chain of inequalities and equalities of (11). First, we show that

$$r_\ell = \max(1, \rho) \quad \text{for } \ell > q. \quad (14)$$

In fact, when  $\ell > q$  the definition of  $F_\ell(x)$  in (13) yields  $F_\ell(x) = x^{\ell-q-1}Q(x)$ , where  $Q(x)$  is the polynomial of (2). Since  $A_\ell = 0$  for  $\ell > q$ , by (12) we have

$$P_\ell(x) \equiv x^{\ell-q-1}(x-1)Q(x), \quad (15)$$

hence the strictly positive zeros of  $P_\ell(x)$  are  $x = 1$  and  $x = \rho$ , thus proving (14).

Finally, we show that  $r_1 \geq r_2 \geq \dots \geq r_q > \max(1, \rho)$ . Let  $\ell \in \{1, \dots, q\}$  be given, and we will see that  $r_{\ell+1} \leq r_\ell$ , with the inequality strict if  $\ell = q$ . Set  $y = r_\ell$ . Since  $0 = P_\ell(y) = (y-1)F_\ell(y) - A_\ell$ , then  $(y-1)F_\ell(y) = A_\ell$ . On the other hand, an easy computation shows that  $F_{\ell+1}(x) = xF_\ell(x) - |a_\ell|$  for all  $x$ . Therefore,

$$\begin{aligned} P_{\ell+1}(y) &= (y-1)F_{\ell+1}(y) - A_{\ell+1} \\ &= (y-1)(yF_\ell(y) - |a_\ell|) - A_{\ell+1} \\ &= y(y-1)F_\ell(y) - (y-1)|a_\ell| - A_{\ell+1} \\ &= yA_\ell - (y-1)|a_\ell| - A_{\ell+1} \\ &= y(A_\ell - |a_\ell|) + |a_\ell| - A_{\ell+1}, \end{aligned} \quad (16)$$

and from  $A_{\ell+1} \leq A_\ell$  we have

$$\begin{aligned} P_{\ell+1}(y) &\geq y(A_\ell - |a_\ell|) + |a_\ell| - A_\ell \\ &= (y-1)(A_\ell - |a_\ell|) \\ &\geq 0. \end{aligned}$$

When  $\ell < q$ , this implies that  $r_\ell = y \geq r_{\ell+1}$ , as we wanted to see. If  $\ell = q$ , from (16) and the fact that  $A_q = |a_q| > 0 = A_{q+1}$ , it follows that  $P_{q+1}(y) = |a_q| > 0$ . Using the expression of  $P_\ell(x)$  in (15), which is valid for  $\ell > q$ , we obtain that  $P_{q+1}(y) = (y-1)Q(y) > 0$ , so  $Q(y) > 0$  since  $y = r_q > 1$ . This implies that  $y > \rho$ , hence  $r_q = y > \max(1, \rho) = r_{q+1}$ , and Theorem 8 follows.  $\square$

Theorem 3 is a straightforward consequence of the following lemma.

**Lemma 10** For any integer number  $\ell \geq 1$ , we have

$$P_\ell(1+x) = Q_\ell(x) - A_\ell, \quad \text{for all } x,$$

where  $Q_\ell(x)$  is the polynomial of (5).

*Proof.* By (12) it is only necessary to see that  $xF_\ell(1+x) = Q_\ell(x)$ . For  $\ell = 1$  it is clear. Set  $b_0 = 1$  and  $b_j = -|a_j|$  for  $j \geq 1$ . Then  $F_\ell(x) = \sum_{k=0}^{\ell-1} b_{\ell-k-1}x^k$  and

$$\begin{aligned} xF_\ell(1+x) &= x \sum_{k=0}^{\ell-1} b_{\ell-k-1}(1+x)^k \\ &= x \sum_{k=0}^{\ell-1} b_{\ell-k-1} \sum_{i=0}^k C_i^k x^i \\ &= \sum_{i=0}^{\ell-1} \left( \sum_{k=i}^{\ell-1} C_i^k b_{\ell-k-1} \right) x^{i+1} \\ &= x^\ell + \sum_{i=0}^{\ell-2} \left( \sum_{k=i}^{\ell-1} C_i^k b_{\ell-k-1} \right) x^{i+1}. \end{aligned}$$

The substitutions  $i = \ell - v$ , ( $2 \leq v \leq \ell$ ), and  $k = \ell - j - 1$ , ( $0 \leq j \leq v - 1$ ), yield

$$\begin{aligned} xF_\ell(1+x) &= x^\ell + \sum_{v=2}^{\ell} \left( \sum_{k=\ell-v}^{\ell-1} C_{\ell-v}^k b_{\ell-k-1} \right) x^{\ell+1-v} \\ &= x^\ell + \sum_{v=2}^{\ell} \left( \sum_{j=0}^{v-1} C_{\ell-v}^{\ell-j-1} b_j \right) x^{\ell+1-v} \\ &= x^\ell + \sum_{v=2}^{\ell} \left( C_{\ell-v}^{\ell-1} - \sum_{j=1}^{v-1} C_{\ell-v}^{\ell-j-1} |a_j| \right) x^{\ell+1-v} \\ &= Q_\ell(x), \end{aligned}$$

and lemma follows.  $\square$

*Proof.* [Proof of Theorem 3] The change of variable  $x = y - 1$ ,  $y \geq 1$ , transforms equation (8) into the equation  $P_\ell(y) = 0$ . Therefore, numbers  $\varepsilon_\ell$  and  $r_\ell$  of Theorems 3 and 8 are related by  $r_\ell = 1 + \varepsilon_\ell$ . Now, applying Theorem 8 we immediately obtain Theorem 3, and this completes the proof.  $\square$

## 4 Conclusion

In this note, we have obtained a wide range of upper bounds for the moduli of the zeros of a complex polynomial. The bounds are summarized in Theorem 3, which completes some other known results on the location of zeros of polynomials.

Finally, we point out that in order to compute the bounds it may be preferable to use Theorem 8 instead of Theorem 3, because it clarifies and simplifies the auxiliary equation to be solved (compare expressions (5) and (10)). For example, when  $\ell = 2$ , it takes the simple form  $x^2 - (|a_1| + 1)x - (A_2 - |a_1|) = 0$ , and it gives rise to the bound by Joyal et al. [5]. For  $\ell = 3$  and  $\ell = 4$ , we obtain

$$x^3 - (|a_1| + 1)x^2 - (|a_2| - |a_1|)x - (A_3 - |a_2|) = 0$$

with  $A_3 = \max_{j \geq 3} |a_j|$ , and

$$x^4 - (|a_1| + 1)x^3 - (|a_2| - |a_1|)x^2 - (|a_3| - |a_2|)x - (A_4 - |a_3|) = 0$$

with  $A_4 = \max_{j \geq 4} |a_j|$ . These equations can be explicitly solved as cubic and quartic equations respectively.

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